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AN APPLICATION OF DYNAMIC PROGRAMMING TO LOCATION-ALLOCATION PROBLEMS

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. This Memorandum indicates a method for treating the problem of minimizing a type of function that frequently arises in scheduling and organization theory. The technique presented here transforms the problem into a dynamic programming one, which can often be readily resolved computationally.

SUMMARY

In a recent paper [1], L. Cooper discusses a number of approaches to the problem of minimizing a function such as

$$f(x,y) = \sum_{i=1}^{N} g_i(x,y).$$

Problems of this nature arise frequently in connection with scheduling and organization theory.

The purpose of this note is to indicate how quasi-linearization may be used to transform this into a dynamic programming problem which in some cases can readily be resolved computationally.

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AN APPLICATION OF DYNAMIC PROGRAMMING TO LOCATION—ALLOCATION PROBLEMS

1. INTRODUCTION

In a recent paper, Cooper [1] discusses a number of approaches to the problem of minimizing a function such as

$$(1.1) \qquad f(x,y) = \sum_{i=1}^{N} g_i(x,y).$$

Problems of this nature arise frequently in connection with scheduling and organization theory.

The purpose of this note is to indicate how quasi-linearization may be used to transform this into a dynamic programming problem which in some cases can readily be resolved computationally.

2. QUASILINEARIZATION

To illustrate the method, let us consider the problem of minimizing

$$(2.1) \qquad f(x,y) = \sum_{i=1}^{N} w_i[(x - a_i)^2 + (y - b_i)^2]^{1/2}, \quad w_i > 0,$$

over all real x and y. This is a particular version of a location problem. (See [1] for a detailed discussion of these questions.) Other references are also to be found there.

We begin with the observation that

$$(2.2)\qquad [(x-a_i)^2+(y-b_i)^2]^{1/2}$$

$$= \max_{u_i^2+v_i^2=1} [(x-a_i)u_i + (y-b_i)v_i].$$

Hence

$$(2.3)\qquad \min_{x,y} f(x,y) = \min_{x,y}\ \max_{u_i,v_i}\ [\sum_i w_i[(x-a_i)u_i + (y-b_i)v_i]]$$

$$= \min_{x,y}\ \max_{u_i,v_i}\ [x\sum_i w_iu_i + y\sum_i w_iv_i - (w_ia_iu_i + w_ib_iv_i)].$$

Using calculus,* we note that the minimizing x and y are determined as solutions of the equations

$$(2.4)\qquad \sum_i \frac{w_i(x-a_i)}{((x-a_i)^2+(y-b_i)^2)^{1/2}} = 0,$$

$$\sum_i \frac{w_i(y-b_i)}{((x-a_i)^2+(y-b_i)^2)^{1/2}} = 0.$$

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*O. Gross points out that in certain cases (2.4) may not hold, since the minimum point $(\bar{x},\bar{y})$ may be at a point of nondifferentiability of a particular $g_i(x,y)$. For example, the points (a_i,b_i) may be $(0,0)$, $(0,1)$, $(1,0)$, $(-1,0)$, $(0,-1)$, with $(0,0)$ the exceptional point. This difficulty can be avoided in a number of ways. Let us assume henceforth that it does not occur.

Since the values of u_i and v_i that maximize in (2.2) are given by

$$u_i = (x - a_i)/(\quad \cdots \quad)^{1/2}, \tag{2.5}$$

$$v_i = (y - b_i)/(\quad \cdots \quad)^{1/2},$$

we see that (2.4) implies that

$$\sum_i w_i u_i = \sum_i w_i v_i = 0 \tag{2.6}$$

for the maximizing u_i and v_i. (Recall also that $u_i^2 + v_i^2 = 1$.) Hence, we may add these conditions to (2.3), obtaining

$$\begin{aligned} \min_{x,y} f(x,y) &= \min_{x,y} \max_{\Sigma w_i u_i = \Sigma w_i v_i = 0} [\sum_i [x \sum_i w_i u_i + y \Sigma w_i v_i \\ &\qquad - w_i(a_i u_i + b_i v_i)]] \\ &= \min_{x,y} \max_{\Sigma w_i u_i = \Sigma w_i v_i = 0} [- w_i(a_i u_i + b_i v_i)] \\ &= \max_{\Sigma w_i u_i = \Sigma w_i v_i = 0} [- w_i(a_i u_i + b_i v_i)], \end{aligned} \tag{2.7}$$

since x and y have now disappeared. We still have the N constraints

$$u_i^2 + v_i^2 = 1, \quad i = 1,2,\ldots,N. \tag{2.8}$$

3. DYNAMIC PROGRAMMING

To obtain a numerical procedure for maximizing

$$- \sum_{i=1}^{N} w_i(a_i u_i + b_i v_i) \tag{3.1}$$

subject to the constraints

$$(3.2)(a) \qquad \sum_{i=1}^{N} w_i u_i = 0,$$

$$(b) \qquad \sum_{i=1}^{N} w_i v_i = 0,$$

$$(c) \qquad u_i^2 + v_i^2 = 1, \quad i = 1,2,\ldots,N,$$

we set

$$(3.3) \qquad u_i = \cos\theta_i, \quad v_i = \sin\theta_i,$$

so that (3.2c) is automatically satisfied. We consider the more general problem of maximizing (3.1) subject to the constraints

$$(3.4) \qquad \sum_{i=1}^{N} w_i \cos\theta_i = c_1,$$

$$\sum_{i=1}^{N} w_i \sin\theta_i = c_2.$$

Setting

$$(3.5) \qquad f_N(c_1,c_2) = \max_{\theta_i} \left[- \sum_{i=1}^{N} w_i(a_i \cos\theta_i + b_i \sin\theta_i) \right],$$

we obtain the recurrence relation

$$(3.6) \qquad f_N(c_1,c_2) = \max_{\theta_N} [- w_N(a_N \cos\theta_N + b_N \sin\theta_N) + f_{N-1}(c_1 - w_N \cos\theta_N, c_2 - w_N \sin\theta_N)],$$

$N \geq 2$, with $f_1(c_1,c_2)$ immediately determined.

Standard procedures can now be used to determine the sequence $\{f_N(c_1,c_2)\}$; see [2].* Ultimately, we want $f_N(0,0)$. There is a small amount of care to be exercised in determining the range of c_1 and c_2 for each N. This can be determined inductively.

4. UPPER AND LOWER BOUNDS

Even if an exact solution is not desired, the transformation to the maximization problem of (2.7) will be useful in furnishing lower bounds.

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*A Lagrange multiplier can be used to reduce the problem to a determination of a sequence of functions of one variable.

REFERENCES

1. Cooper, L., "Heuristic Methods for Location-Allocation Problems," SIAM Review, Vol. 6, 1964, pp. 37–53.

2. Bellman, R., and S. Dreyfus, Applied Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1962.